

MARTINGALE INEQUALITIES IN VARIABLE EXPONENT HARDY SPACES WITH $0 < p^- \leq p^+ < \infty$

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ABSTRACT. We investigate the properties of the variable Lebesgue spaces with quasi-norm on a probability space, and give the atomic decompositions suited to the variable exponent martingale Hardy spaces. Using the decompositions and the harmonic mean of a variable exponent, we obtain several continuous embedding relations between martingale Hardy spaces with small exponent. Finally, we extend these results to the cases $0 < p^- \leq p^+ < \infty$.

1. INTRODUCTION

Variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ is defined as the set of all measurable functions f such that for some $\lambda > 0$

$$\int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty,$$

where $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$ is a measurable function. These spaces were introduced by Orlicz [22] in 1931. The variable Lebesgue spaces, as their name implies, are a generalization of the classical Lebesgue spaces, replacing the constant exponent p with a variable exponent function $p(\cdot)$. The $L^{p(\cdot)}(\mathbb{R}^n)$ spaces have many properties similar to the classical $L^p(\mathbb{R}^n)$ spaces, but they also differ from each other in surprising and subtle ways. For this reason the variable Lebesgue spaces have an intrinsic interest. In addition, they are also very important for their applications to PDEs, variational integrals with nonstandard local growth conditions, non-Newtonian fluids and image restoration. In the past few years the subject of variable exponent spaces has undergone a vast development (see [6, 9] for the history and references). Recently, the theory of variable Lebesgue spaces was extended to that of variable Hardy spaces. The variable Hardy spaces had been developed independently by Nakai and Sawano [21] and Cruz-Uribe and Daniel Wang [5]. They defined atomic decompositions and proved the equivalent definitions in terms of maximal operators. Then they gave that singular integrals are bounded on variable Hardy spaces.

As is well known, martingale space theory is very closely related to harmonic analysis and functional space theory. Over the past few years, some authors have paid attention to variable exponent martingale spaces and variable exponent martingale Hardy spaces. In

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particular, Nakai and Sadasue [20] studied the boundedness of Doob's maximal operator on variable exponent martingale spaces. Jiao, Zhou, Hao and Chen [13] investigated the atomic decompositions and John-Nirenberg inequalities for variable exponent martingale Hardy spaces. In [15], the famous Burkholder-Gundy-Davis inequality for martingales and some continuous embedding relations between martingale spaces in classical martingale theory were extended to variable exponent Hardy spaces. We mention that variable exponent martingale (Hardy) spaces are very different from classical martingale spaces and function spaces. For example, it is clear that the Jensen's inequality for the conditional expectation is invalid. Moreover, the log-Hölder continuity (see, e.g. [6, 9]) is also invalid, because the probability space Ω has no natural metric structure and linear structure. Thus, it is difficult but interesting to study variable exponent martingale (Hardy) spaces.

The aim of this paper is to deal with variable exponent martingale (Hardy) spaces. In classical martingale theory, Burkholder-Gundy-Davis inequality holds only for any $1 \leq p < \infty$, but some other inequalities hold for all $0 < p < \infty$, such as several inequalities for the martingales with predictable controls (see [16, 24] for more information). Our goal is to extend the latter to the case $0 < p^- \leq p^+ < \infty$ and our approaches are mainly based on the atomic decompositions suited to the variable exponent martingale Hardy spaces.

The article is organized as follows. In section 2, we investigate some basic properties of variable exponent Lebesgue space $L^{p(\cdot)}$ with exponent $0 < p^- \leq p^+ < \infty$. In section 3, we establish several atomic decomposition theorems for the martingale Hardy spaces $H_{p(\cdot)}^s$, $\mathcal{D}_{p(\cdot)}$ and $\mathcal{Q}_{p(\cdot)}$. In section 4, we discuss some properties of harmonic mean of a variable exponent. In section 5, using the atomic decompositions and the harmonic mean, we obtain several martingale inequalities and continuous embedding relations between the spaces with small exponents. In the last section, we extend the theorems in section 5 to the cases $0 < p^- \leq p^+ < \infty$.

Throughout this paper, \mathbb{Z} denotes the integer set. We denote by C the absolute positive constant, which can vary from line to line, and denote by $C_{p(\cdot)}$ the constant depending only on $p(\cdot)$. The symbol $A \lesssim B$ stands for the inequality $A \leq CB$ or $A \leq C_{p(\cdot)}B$. If we write $A \sim B$, then it stands for $A \lesssim B \lesssim A$.

2. ON VARIABLE LEBESGUE SPACES WITH QUASI-NORM

Let (Ω, Σ, μ) be a complete probability space. We denote by $L^0(\Omega)$ the set of all measurable functions on Ω and $L_+^0(\Omega)$ the set of all positive members in $L^0(\Omega)$. For $p \in L_+^0(\Omega)$, we call p a variable exponent. Let $p^- = \text{ess inf}_{\omega \in \Omega} p(\omega)$ and $p^+ = \text{ess sup}_{\omega \in \Omega} p(\omega)$. If $0 < p^- \leq p^+ < \infty$, we say $p \in \mathcal{P}$. For a variable exponent p , we define the modular of $u \in L^0(\Omega)$ by

$$(2.1) \quad \rho_{p(\cdot)}(u) = E(|u|^p \chi_{\{p < \infty\}}) + \|u \chi_{\{p = \infty\}}\|_\infty,$$

where E is the expectation with respect to Σ . Then, we denote the variable exponent Lebesgue space by

$$L^{p(\cdot)} = \{u \in L^0(\Omega) : \exists \gamma > 0, \rho_{p(\cdot)}(\gamma u) < \infty\}$$

with (quasi-)norm

$$(2.2) \quad \|u\|_{p(\cdot)} = \inf\{\gamma > 0 : \rho_{p(\cdot)}(\frac{u}{\gamma}) \leq 1\}, \quad \forall u \in L^{p(\cdot)}.$$

In the rest of this section we state some basic properties of $L^{p(\cdot)}$ (see [5, 6, 9]). For convenience, we give their proofs.

Lemma 2.1. *Let $p \in \mathcal{P}$ and $u \in L^{p(\cdot)}$.*

- (1) $\rho_{p(\cdot)}(\lambda u)$ is continuous with respect to λ on $(0, \infty)$;
- (2) $\rho_{p(\cdot)}(u) < 1$ ($= 1, > 1$) iff $\|u\|_{p(\cdot)} < 1$ ($= 1, > 1$);
- (3) if $\|u\|_{p(\cdot)} \leq 1$, then $\|u\|_{p(\cdot)}^{p^+} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^-}$;
- (4) if $\|u\|_{p(\cdot)} \geq 1$, then $\|u\|_{p(\cdot)}^{p^+} \geq \rho_{p(\cdot)}(u) \geq \|u\|_{p(\cdot)}^{p^-}$.

Proof of Lemma 2.1. It is clear that the function $|\lambda u|^p$ is increasing and continuous with respect to λ . In view of Levy's theorem and Lebesgue's dominated convergence theorem, we get $\rho_{p(\cdot)}$'s left-continuity and right-continuity, respectively. Then, we have (1). By the definition of $\|u\|_{p(\cdot)}$ and (1), we obtain (2).

To prove (3), let $0 < \|u\|_{p(\cdot)} \leq 1$. Then we have

$$\frac{\rho(u)}{\|u\|_{p(\cdot)}^{p^-}} = E\left(\frac{|u|^p}{\|u\|_{p(\cdot)}^{p^-}}\right) \leq E\left(\frac{u}{\|u\|_{p(\cdot)}}\right)^p \leq E\left(\frac{|u|^p}{\|u\|_{p(\cdot)}^{p^+}}\right) = \frac{\rho(u)}{\|u\|_{p(\cdot)}^{p^+}}.$$

It follows from (2) that $E(\frac{u}{\|u\|_{p(\cdot)}})^p = \rho_{p(\cdot)}(\frac{u}{\|u\|_{p(\cdot)}}) = 1$. Thus $\|u\|_{p(\cdot)}^{p^+} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^-}$. Similarly, we have (4). \square

Lemma 2.2. *Let $p \in \mathcal{P}$, $u_n, u \in L^{p(\cdot)}$.*

- (1) $\sup_n \|u_n\|_{p(\cdot)} < \infty$ iff $\sup_n \rho_{p(\cdot)}(u_n) < \infty$;
- (2) $\|u_n - u\|_{p(\cdot)} \rightarrow 0$ iff $\rho_{p(\cdot)}(u_n - u) \rightarrow 0$;
- (3) $L^{p(\cdot)}$ is a quasi-Banach space;
- (4) if $p^- \geq 1$, $L^{p(\cdot)}$ is a Banach space.

Proof of Lemma 2.2. To prove (1). Let $\sup_n \|u_n\|_{p(\cdot)} < \infty$. In view of Lemma 2.1(3) and (4), we have

$$\rho_{p(\cdot)}(u_n) \leq \begin{cases} \|u_n\|_{p(\cdot)}^{p^-}, & \|u_n\|_{p(\cdot)} \leq 1; \\ \|u_n\|_{p(\cdot)}^{p^+}, & \|u_n\|_{p(\cdot)} \geq 1. \end{cases}$$

Then

$$\rho_{p(\cdot)}(u_n) \leq (\sup_n \|u_n\|_{p(\cdot)})^{p^-} + (\sup_n \|u_n\|_{p(\cdot)})^{p^+} < \infty.$$

Let $\rho_{p(\cdot)}(u_n) < \infty$. In the view of Lemma 2.1(3) and (4), we have

$$\|u_n\|_{p(\cdot)} \leq \begin{cases} \rho_{p(\cdot)}(u_n)^{\frac{1}{p^+}}, & \|u_n\|_{p(\cdot)} \leq 1; \\ \rho_{p(\cdot)}(u_n)^{\frac{1}{p^-}}, & \|u_n\|_{p(\cdot)} \geq 1. \end{cases}$$

Then

$$\|u_n\|_{p(\cdot)} \leq (\sup_n \rho_{p(\cdot)}(u_n))^{\frac{1}{p^+}} + (\sup_n \rho_{p(\cdot)}(u_n))^{\frac{1}{p^-}} < \infty.$$

It is clear that (2) follows directly from Lemma 2.1(3).

To check that $\|\cdot\|_{p(\cdot)}$ is a quasi-norm. By the definition of $\|u\|_{p(\cdot)}$, we have $\|u\|_{p(\cdot)} \geq 0$ and $\|u\|_{p(\cdot)} = 0$ iff $\rho_{p(\cdot)}(u) = 0$ iff $u = 0$. Let $\alpha \neq 0$. It follows that

$$\|\alpha u\|_{p(\cdot)} = \inf\{\gamma > 0 : \rho(\frac{\alpha u}{\gamma}) \leq 1\} = |\alpha| \inf\{\frac{\gamma}{|\alpha|} > 0 : \rho(\frac{\alpha u}{\gamma}) \leq 1\} = |\alpha| \|u\|_{p(\cdot)}.$$

Now if $0 < \|u\|_{p(\cdot)} < a$, $0 < \|v\|_{p(\cdot)} < b$ and $K \geq 1$, we deduce that

$$\begin{aligned} \rho_{p(\cdot)}(\frac{u+v}{K(a+b)}) &= E(\frac{(u+v)}{K(a+b)})^p \leq E(\frac{2}{K} \max\{|\frac{u}{a}|, |\frac{v}{b}|\})^p \\ &\leq \frac{2^{p^+}}{K^{p^-}} (E|\frac{u}{a}|^p + E|\frac{v}{b}|^p) \leq \frac{2^{p^++1}}{K^{p^-}} \leq 1, \end{aligned}$$

provided K is large enough. Then $\|u+v\|_{p(\cdot)} \leq K(a+b)$. Finally we get

$$(2.3) \quad \|u+v\|_{p(\cdot)} \leq K(\|u\|_{p(\cdot)} + \|v\|_{p(\cdot)}).$$

The proof of the completeness of $\|\cdot\|_{p(\cdot)}$ is an easy modification of the standard one, and we omit it.

The proof of (4) is well known: see, for instance, [6, Theorem 2.70]. \square

Lemma 2.3. (1) Let $p \in \mathcal{P}$, $u \in L^{p(\cdot)}$ and $0 < s < \infty$. Then

$$(2.4) \quad \| |u|^s \|_{p(\cdot)} = \|u\|_{sp(\cdot)}^s.$$

(2) If $r, p, q \in \mathcal{P}$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then there is $C = C_{p,q} > 0$ such that

$$(2.5) \quad \|uv\|_{r(\cdot)} \leq C \|u\|_{p(\cdot)} \|v\|_{q(\cdot)}, \quad \forall u \in L^{p(\cdot)}, v \in L^{q(\cdot)}.$$

Proof of Lemma 2.3. To prove (1). Since $p^- s > 0$, we have

$$\begin{aligned} \|u\|_{sp(\cdot)}^s &= \inf\{\gamma^s > 0 : E|\frac{u}{\gamma}|^{sp} \leq 1\} \\ &= \inf\{\gamma^s > 0 : E(\frac{|u|^s}{\gamma^s})^p \leq 1\} = \| |u|^s \|_{p(\cdot)}. \end{aligned}$$

To prove (2). Without loss of generality, we suppose that $\|u\|_{p(\cdot)} = \|v\|_{q(\cdot)} = 1$. Then we have

$$E(|u|^r)^{\frac{p}{r}} = E|u|^p = 1, \quad E(|v|^r)^{\frac{q}{r}} = E|v|^q = 1.$$

It follows that $\| |u|^r \|_{\frac{p(\cdot)}{r(\cdot)}} = \| |v|^r \|_{\frac{q(\cdot)}{r(\cdot)}} = 1$. Since $\frac{p(\cdot)}{r(\cdot)} > 1$, $\frac{q(\cdot)}{r(\cdot)} > 1$ and $\frac{r(\cdot)}{p(\cdot)} + \frac{r(\cdot)}{q(\cdot)} = 1$, we get

$$\rho_{r(\cdot)}(uv) = E(|u||v|)^r \leq C \| |u|^r \|_{\frac{p(\cdot)}{r(\cdot)}} \| |v|^r \|_{\frac{q(\cdot)}{r(\cdot)}} = C$$

by Hölder's inequality in variable exponent case([6, Lemma 2.2.26]). In view of Lemma 2.1(3), we have $\|uv\|_{r(\cdot)} \leq C = C \|u\|_{p(\cdot)} \|v\|_{q(\cdot)}$. \square

Lemma 2.4. Let $p \in \mathcal{P}$ with $p^+ \leq 1$ and $u \in L^{p(\cdot)}$. Then

(1) $\rho_{p(\cdot)}(u)$ is concave: $\forall u, v \in L^{p(\cdot)}, \alpha, \beta \geq 0, \alpha + \beta = 1$,

$$\rho_{p(\cdot)}(\alpha u + \beta v) \geq \alpha \rho_{p(\cdot)}(u) + \beta \rho_{p(\cdot)}(v).$$

(2) if $\|u\|_{p(\cdot)} \leq 1$, then $\|u\|_{p(\cdot)} \leq \rho_{p(\cdot)}(u)$; if $\|u\|_{p(\cdot)} \geq 1$, then $\|u\|_{p(\cdot)} \geq \rho_{p(\cdot)}(u)$.

Proof of Lemma 2.4. For any α, β, u, v , since $p \leq 1$, we have $|\alpha u + \beta v|^p \geq \alpha|u|^p + \beta|v|^p$. It follows that

$$\rho_{p(\cdot)}(\alpha u + \beta v) \geq \alpha \rho_{p(\cdot)}(u) + \beta \rho_{p(\cdot)}(v).$$

In view of Lemma 2.1(3), we get (2). \square

In this paper, we shall use the following theorem many times, which is known as Aoki-Rolewicz's theorem.

Theorem 2.5. [18, Theorem 1.3] *Let X be a vector space equipped with a quasinorm $\|\cdot\|$. Then there exists a quasinorm $\|\cdot\|_*$ on X that is equivalent to $\|\cdot\|$ and is a η -norm for some $0 < \eta \leq 1$, i.e., it satisfies $\|x + y\|_*^\eta \leq \|x\|_*^\eta + \|y\|_*^\eta$ for all $x, y \in X$.*

Remark 2.6. If $p^+ < 1$, denote $\|\cdot\|_{p(\cdot)} = \|\cdot\|_*$, then we have $\|\cdot\|_{p(\cdot)} \approx \|\cdot\|_{p(\cdot)}^\eta$ and

$$(2.6) \quad \|\|u + v\|\|_{p(\cdot)} \leq \|\|u\|\|_{p(\cdot)} + \|\|v\|\|_{p(\cdot)}, \quad \forall u, v \in L^{p(\cdot)}.$$

3. THE ATOMIC DECOMPOSITIONS

Let $(\Sigma_n)_{n \geq 0}$ be a nondecreasing sequence of sub- σ -algebras of Σ with $\Sigma = \bigvee \Sigma_n$ and $f = (f_n)_{n \geq 0}$ a martingale adapted to $(\Sigma_n)_{n \geq 0}$ with its difference sequence $(d_n f)_{n \geq 0}$, where $d_n f = f_n - f_{n-1}$ (with convention $f_{-1} = 0, \Sigma_{-1} = \{\Omega, \emptyset\}$). We denote by E_n the conditional expectation with respect to Σ_n . For a martingale $f = (f_n)_{n \geq 0}$, we define its maximal function f^* , square function $S(f)$ and conditional square function $s(f)$ as usual. The variable exponent Hardy spaces of martingales are defined as follows:

$$\begin{aligned} H_{p(\cdot)}^* &= \{f = (f_n) : \|f\|_{H_{p(\cdot)}^*} = \|f^*\|_{p(\cdot)} < \infty\}, \\ H_{p(\cdot)}^S &= \{f = (f_n) : \|f\|_{H_{p(\cdot)}^S} = \|S(f)\|_{p(\cdot)} < \infty\}, \\ H_{p(\cdot)}^s &= \{f = (f_n) : \|f\|_{H_{p(\cdot)}^s} = \|s(f)\|_{p(\cdot)} < \infty\}. \end{aligned}$$

Let Λ be the class of non-negative, non-decreasing and adapted sequence $\lambda = (\lambda_n)$ with $\lambda_\infty = \lim_{n \rightarrow \infty} \lambda_n$. We define

$$\begin{aligned} \mathcal{Q}_{p(\cdot)} &= \{f = (f_n) : \exists \lambda \in \Lambda, S_n(f) \leq \lambda_{n-1}, \|f\|_{\mathcal{Q}_{p(\cdot)}} = \inf_{\lambda \in \Lambda} \|\lambda_\infty\|_{p(\cdot)} < \infty\}, \\ \mathcal{D}_{p(\cdot)} &= \{f = (f_n) : \exists \lambda \in \Lambda, |f_n| \leq \lambda_{n-1}, \|f\|_{\mathcal{D}_{p(\cdot)}} = \inf_{\lambda \in \Lambda} \|\lambda_\infty\|_{p(\cdot)} < \infty\}. \end{aligned}$$

Using the method of [16, P.51], we can construct a $\lambda \in \Lambda$ such that

$$\|f\|_{\mathcal{Q}_{p(\cdot)}} = \|\lambda_\infty\|_{p(\cdot)} \text{ or } \|f\|_{\mathcal{D}_{p(\cdot)}} = \|\lambda_\infty\|_{p(\cdot)},$$

which is called f 's optimal predictable control in $\mathcal{Q}_{p(\cdot)}$ or $\mathcal{D}_{p(\cdot)}$, respectively.

Let $p \in \mathcal{P}$. We say that a Σ -measurable function a is an atom of first category, if there exists a stopping time τ such that

- (1) $E_n a = 0$, when $n \leq \tau$;
- (2) $\|s(a)\|_\infty \leq \|\chi_{\{\tau < \infty\}}\|_{p(\cdot)}^{-1}$.

a is said to be an atom of second or third category if (2) holds when we use $S(a)$ or a^* instead of $s(a)$, respectively. We denote by $p - A_1, p - A_2$ or $p - A_3$ the sets of all atoms of first, second, or third category, respectively.

The atomic decomposition of Hardy spaces of functions defined on R^n is due to Coifman [3] (see [4] for more information). Meanwhile, Herz [12] introduced the atomic decomposition to martingale theory. Then Bernard and Maisonneuve [1], Chevalier [2], Weisz [23] used them to studied martingale space theory (see [24] for more information). In this section, we establish some atomic decomposition theorems for the martingales in $H_{p(\cdot)}^s, \mathcal{Q}_{p(\cdot)}$ and $\mathcal{D}_{p(\cdot)}$, respectively.

Theorem 3.1. *Let $p \in \mathcal{P}$ and $f \in H_{p(\cdot)}^s$. Then there exist a sequence $(a^k, k \in Z)$ of $p - A_1$ atoms with $\|a^k\|_{H_{p(\cdot)}^s} \leq 1$, and a sequence $(\theta_k, k \in Z)$ of nonnegative numbers such that*

$$(3.1) \quad f_n = \sum_{k \in Z} \theta_k E_n a^k, \text{ a.e. } \forall n \geq 0$$

and

$$(3.2) \quad C^{-1} \left(\sum_{k \in Z} \theta_k^{p^+} \right)^{\frac{1}{p^+}} \leq \|f\|_{H_{p(\cdot)}^s} \leq C \left(\sum_{k \in Z} \theta_k^{p^-} \right)^{\frac{1}{p^-}},$$

where C is a constant independent of f .

Moreover, the sum $\sum_{k=j}^m \theta_k a^k$ converges to f in $H_{p(\cdot)}^s$ as $j \rightarrow -\infty, m \rightarrow \infty$.

Proof of Theorem 3.1. For $f = (f_n) \in H_{p(\cdot)}^s$ and $k \in Z$, define stopping times

$$(3.3) \quad \tau_k = \inf \{n : s_{n+1}(f) > 2^k\}, \quad (\inf \emptyset = \infty)$$

and $\theta_k = 3 \cdot 2^k \|\chi_{A_k}\|_{p(\cdot)}$, where $A_k = \{\tau_k < \infty\} = \{s(f) > 2^k\}$. Let

$$a_n^k = \theta_k^{-1} (f_{n_{\tau_k+1}} - f_n^{\tau_k}), \quad a^k = (a_n^k)_{n=0}^\infty.$$

Then $da_n^k = \theta_k^{-1} (df_{\tau_{k+1} \wedge n} - df_{\tau_k \wedge n})$. It follows that a^k is a martingale with

$$(3.4) \quad E_n a^k = 0, \quad \forall n \leq \tau_k$$

and

$$(3.5) \quad \sum_{k \in Z} \theta_k a_n^k = \sum_{k \in Z} (f_{\tau_{k+1} \wedge n} - f_{\tau_k \wedge n}) = f_n.$$

Moreover

$$s(a^k) \leq \theta_k^{-1} (s_{\tau_{k+1}}(f) + s_{\tau_k}(f)) \leq \theta_k^{-1} (3 \cdot 2^k) \leq \|\chi_{A_k}\|_{p(\cdot)}^{-1}.$$

Thus

$$(3.6) \quad \|s(a^k)\|_\infty \leq \|\chi_{A_k}\|_{p(\cdot)}^{-1}, \quad \forall k \in Z.$$

By classical Burkholder-Gundy-Davis inequality, a_n^k converges to a function a.e. and in L_2 . We still denote the function by a^k . Then $a^k \in L_2$ and $a_n^k = E_n a^k$. It follows from (3.4) and (3.6) that every a^k is a $p - A_1$ atom and (3.1) holds. In addition, using (3.6), we get

$$\rho_{p(\cdot)}(s(a^k)) = E s(a^k)^p \chi_{A_k} = E [s(a^k) \|\chi_{A_k}\|_{p(\cdot)}]^p \left[\frac{\chi_{A_k}}{\|\chi_{A_k}\|_{p(\cdot)}} \right]^p \leq E \left[\frac{\chi_{A_k}}{\|\chi_{A_k}\|_{p(\cdot)}} \right]^p = 1,$$

Then, by Lemma 2.1(2), we have

$$\|a^k\|_{H_{p(\cdot)}^s} = \|s(a^k)\|_{p(\cdot)} \leq 1, \quad \forall k \in Z.$$

To estimate (3.2), we split it into three steps.

Step 1. We define the function $g = \sum_{k \in Z} 3 \cdot 2^k \chi_{A_k}$ and show $\|g\|_{p(\cdot)} \sim \|f\|_{H_{p(\cdot)}^s}$. It is clear that

$$g = 3 \sum_{k \in Z} 2^k \chi_{A_k} = 3 \sum_{k \in Z} (2^{k+1} - 2^k) \chi_{A_k} = 6 \sum_{k \in Z} 2^k \chi_{A_k \setminus A_{k+1}}.$$

For any $\gamma > 0$, we have

$$\begin{aligned} \int_{\Omega} \left(\frac{\sum_{k \in Z} 2^k \chi_{A_k \setminus A_{k+1}}}{\gamma} \right)^p d\mu &= \sum_{k \in Z} \int_{A_k \setminus A_{k+1}} \left(\frac{2^k}{\gamma} \right)^p d\mu \\ &\leq \sum_{k \in Z} \int_{A_k \setminus A_{k+1}} \left(\frac{s(f)}{\gamma} \right)^p d\mu \leq \int_{\Omega} \left(\frac{s(f)}{\gamma} \right)^p d\mu. \end{aligned}$$

Taking $\gamma = \|s(f)\|_{p(\cdot)}$ and noticing $\int_{\Omega} \left(\frac{s(f)}{\gamma} \right)^p d\mu = \rho_{p(\cdot)} \left(\frac{s(f)}{\|s(f)\|_{p(\cdot)}} \right) = 1$, we get

$$\left\| \sum_{k \in Z} 2^k \chi_{A_k \setminus A_{k+1}} \right\|_{p(\cdot)} \leq \gamma.$$

Hence $\|g\|_{p(\cdot)} \leq 6\|f\|_{H_{p(\cdot)}^s}$. On the other hand, for any $\gamma > 0$, we have

$$\int_{\Omega} \left(\frac{\sum_{k \in Z} 2^{k+1} \chi_{A_k \setminus A_{k+1}}}{\gamma} \right)^p d\mu = \sum_{k \in Z} \int_{A_k \setminus A_{k+1}} \left(\frac{2^{k+1}}{\gamma} \right)^p d\mu \geq \int_{\Omega} \left(\frac{s(f)}{\gamma} \right)^p d\mu,$$

Taking $\gamma = \|s(f)\|_{p(\cdot)}$, we get

$$\left\| \sum_{k \in Z} 2^{k+1} \chi_{A_k \setminus A_{k+1}} \right\|_{p(\cdot)} \geq \gamma.$$

Hence $\|g\|_{p(\cdot)} \geq 3\|f\|_{H_{p(\cdot)}^s}$.

Step 2. To estimate the first inequality in (3.2). For any $\gamma > 0$, we have

$$\begin{aligned} E \left(\frac{g}{\gamma} \right)^p &= E \left(\sum_{k \in Z} \frac{6 \cdot 2^k \chi_{A_k \setminus A_{k+1}}}{\gamma} \right)^p = E \sum_{k \in Z} \left(\frac{6 \cdot 2^k \chi_{A_k \setminus A_{k+1}}}{\gamma} \right)^p \\ &= E \sum_{k \in Z} (2^p - 1) \left(\frac{3 \cdot 2^k \chi_{A_k}}{\gamma} \right)^p \geq (2^{p^-} - 1) E \sum_{k \in Z} \left(\frac{\theta_k \chi_{A_k}}{\gamma \|\chi_{A_k}\|_{p(\cdot)}} \right)^p. \end{aligned}$$

Taking $\gamma = (\sum_{k \in Z} \theta_k^{p^+})^{\frac{1}{p^+}}$ and denoting $C = 2^{p^-} - 1$, we get

$$E \left(\frac{g}{\gamma} \right)^p \geq C \sum_{k \in Z} \left(\frac{\theta_k}{\gamma} \right)^{p^+} E \left(\frac{\chi_{A_k}}{\|\chi_{A_k}\|_{p(\cdot)}} \right)^p = C \sum_{k \in Z} \left(\frac{\theta_k}{\gamma} \right)^{p^+} = C.$$

Lemma 2.1(3) shows that $\|g\|_{p(\cdot)} \geq C$. It follows that $\|g\|_{p(\cdot)} \geq C\gamma = C(\sum_{k \in Z} \theta_k^{p^+})^{\frac{1}{p^+}}$.

Step 3. To estimate the second inequality in (3.2). For any $\gamma > 0$, we have

$$E\left(\frac{g}{\gamma}\right)^p = E \sum_{k \in Z} (2^p - 1) \left(\frac{3 \cdot 2^k \chi_{A_k}}{\gamma}\right)^p \leq (2^{p^+} - 1) E \sum_{k \in Z} \left(\frac{\theta_k \chi_{A_k}}{\gamma \|\chi_{A_k}\|_{p(\cdot)}}\right)^p.$$

If $\sum_{k \in Z} \theta_k^{p^-} < \infty$, taking $\gamma = (\sum_{k \in Z} \theta_k^{p^-})^{\frac{1}{p^-}}$ (the inequality naturally holds when $\sum_{k \in Z} \theta_k^{p^-} = \infty$) and denoting $C = 2^{p^+} - 1$, we get

$$E\left(\frac{g}{\gamma}\right)^p \leq C \sum_{k \in Z} \left(\frac{\theta_k}{\gamma}\right)^{p^-} E\left(\frac{\chi_{A_k}}{\|\chi_{A_k}\|_{p(\cdot)}}\right)^p = C \sum_{k \in Z} \left(\frac{\theta_k}{\gamma}\right)^{p^-} = C.$$

Therefore,

$$\|g\|_{p(\cdot)} \leq C\gamma = C\left(\sum_{k \in Z} \theta_k^{p^-}\right)^{\frac{1}{p^-}}.$$

Now we prove that the sum $\sum_{k=j}^m \theta_k a^k$ converges to f in $H_{p(\cdot)}^s$. It follows from (3.1) that $f = \sum_{k \in Z} \theta_k a^k$ and $f - \sum_{k=j}^m \theta_k a^k = f - f^{\tau_{m+1}} + f^{\tau_j}$. Because of $s(f^{\tau_j}) \leq 2^j$, we have $\|f^{\tau_j}\|_{p(\cdot)} \leq 2^j \rightarrow 0$ as $j \rightarrow -\infty$. In addition, we have

$$s(f - f^{\tau_{m+1}})^p \leq s(f)^p \text{ and } s(f - f^{\tau_{m+1}})^p \rightarrow 0.$$

In view of Lebesgue's dominated convergence, we have $\rho_{p(\cdot)}(s(f - f^{\tau_{m+1}})) \rightarrow 0$ as $m \rightarrow \infty$. It follows that $\|s(f - f^{\tau_{m+1}})\|_{p(\cdot)} \rightarrow 0$ as $m \rightarrow \infty$. Finally, the sum $\sum_{k=j}^m \theta_k a^k$ converges to f in $H_{p(\cdot)}^s$ as $j \rightarrow -\infty, m \rightarrow \infty$. \square

Theorem 3.2. *Let $p \in \mathcal{P}$ and $f \in \mathcal{Q}_{p(\cdot)}(\mathcal{D}_{p(\cdot)})$. Then there exist a sequence $(a^k, k \in Z)$ of $p - A_2(p - A_3)$ atoms and a sequence $(\theta_k, k \in Z)$ of nonnegative numbers such that*

$$(3.7) \quad f_n = \sum_{k \in Z} \theta_k E_n a^k, \text{ a.e. } \forall n \geq 0$$

and

$$(3.8) \quad C^{-1} \left(\sum_{k \in Z} \theta_k^{p^+}\right)^{\frac{1}{p^+}} \leq \|f\|_{\mathcal{Q}_{p(\cdot)}} (\|f\|_{\mathcal{D}_{p(\cdot)}}) \leq C \left(\sum_{k \in Z} \theta_k^{p^-}\right)^{\frac{1}{p^-}},$$

where C is a constant independent of f .

Moreover, if $p^+ \leq \eta$, then $\sum_{k=j}^m \theta_k a^k$ converges to f in $\mathcal{Q}_{p(\cdot)}(\mathcal{D}_{p(\cdot)})$'s quasi-norm, as $j \rightarrow -\infty, m \rightarrow \infty$.

Proof of Theorem 3.2. The proof is similar to that of Theorem 3.1. Here we only give a outline of the proof for $\mathcal{Q}_{p(\cdot)}$.

Let $f \in \mathcal{Q}_{p(\cdot)}$ and $\lambda = (\lambda_n)$ be $S_n(f)$'s optimal predictable control. We define

$$\tau_k = \inf\{n : \lambda_n > 2^k\}, \quad A_k = \{\tau_k < \infty\} = \{\lambda_\infty > 2^k\}$$

and θ_k, a_n^k, a^k as in the proof of Theorem 3.1. Instead of estimating $s(f)$, we estimate λ_∞ . We have

$$\|\lambda_\infty\|_\infty \leq \|\chi_{A_k}\|_{p(\cdot)}^{-1}, \quad \|a^k\|_{\mathcal{Q}_{p(\cdot)}} \leq 1$$

and (3.7) holds. To estimate $\|\lambda_\infty\|_{p(\cdot)}$, we define g as in the proof of Theorem 3.1. Then similar computations show that $\|\lambda_\infty\|_{p(\cdot)} \sim \|g\|_{p(\cdot)}$ and (3.8) holds.

It remains to prove the last statement. It is clear that

$$S(f - f^{\tau_{m+1}}) \leq \sum_{k \geq m+1} S(f^{\tau_{k+1}} - f^{\tau_k}) = \sum_{k \geq m+1} \theta_k S(a^k).$$

Set $\rho_n^k = \|\lambda_\infty\|_{p(\cdot)} \chi_{\{\tau_k \leq n\}}$, $\rho_n = \sum_{k=m+1}^\infty \theta_k \rho_n^k$, then ρ_n^k is adapted and

$$S_n(f - f^{\tau_{m+1}}) \leq \sum_{k \geq m+1} S(f_n^{\tau_{k+1}} - f_n^{\tau_k}) \leq \sum_{k \geq m+1} \theta_k S_n(a^k) \leq \rho_{n-1}.$$

It follows that $(\rho_n, n \geq 0)$ is a predictable control of $(S_n(f - f^{\tau_{m+1}}), n \geq 0)$. In addition, we have

$$\begin{aligned} \|f - f^{\tau_{m+1}}\|_{\mathcal{Q}_{p(\cdot)}}^\eta &\leq \|\rho_\infty\|_{p(\cdot)}^\eta \leq C \sum_{k \geq m+1} \|\theta_k \lambda_\infty \chi_{A_k}\|_{p(\cdot)}^\eta \\ &\leq C \sum_{k \geq m+1} \theta_k^\eta \leq C \left(\sum_{k \geq m+1} \theta_k^{p^+} \right)^{\frac{\eta}{p^+}}. \end{aligned}$$

Thus, $\|f - f^{\tau_{m+1}}\|_{\mathcal{Q}_{p(\cdot)}} \rightarrow 0$ in as $m \rightarrow +\infty$. Moreover, because of $S(f^{\tau_j}) \leq 2^j$, we have that $\|f^{\tau_j}\|_{\mathcal{Q}_{p(\cdot)}} \rightarrow 0$ as $j \rightarrow -\infty$. Finally, $\sum_{k=j}^m \theta_k a^k$ converges to f in $\mathcal{Q}_{p(\cdot)}$, as $j \rightarrow -\infty, m \rightarrow \infty$. \square

4. ON HARMONIC MEAN OF VARIABLE EXPONENT

In this section we define the harmonic mean and the averaging operator. Using the method of proving [9, Lemma 4.5.3.], we estimate $\|\chi_A\|_{p(\cdot)}$, where $A \in \Sigma$ and $p \in \mathcal{P}$. In what follows, $\mu(A)$ is denoted by $|A|$, $\forall A \in \Sigma$.

Definition 4.1. [9] Let $p \in L_+^0(\Omega)$ and $A \in \Sigma$ with $|A| > 0$. We define the harmonic mean of p by setting

$$\frac{1}{p_A} = \frac{1}{|A|} \int_A \frac{1}{p} d\mu.$$

By Definition 4.1, we immediately deduce Lemma 4.2.

Lemma 4.2. Let $p, q, r \in L_+^0(\Omega)$.

- (1) If $s > 0$, then $(sp)_A = sp_A$.
- (2) If $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then $\frac{1}{r_A} = \frac{1}{p_A} + \frac{1}{q_A}$.

Definition 4.3. Let $A \in \Sigma$ with $|A| > 0$. We define the averaging operator $T_A : L^1 \rightarrow L^0$ by setting $T_A u = \frac{1}{|A|} \int_A u d\mu \chi_A$.

Lemma 4.4. If $1 \leq p^- \leq p^+ < \infty$, the operator $T_A : L^{p(\cdot)} \rightarrow L^{p(\cdot)}$ is uniformly bounded with respect to $A \in \Sigma$.

Proof of Lemma 4.4. Recall that there is a constant $C > 0$ such that

$$(4.1) \quad \sup_{\lambda > 0} \|\lambda \chi_{\{f^* > \lambda\}}\|_{p(\cdot)} \leq C \|f\|_{p(\cdot)}, \quad \forall f = (f_n),$$

where f^* is Doob's maximal operator (see [13]).

For $u \in L^{p(\cdot)}$ and $A \in \Sigma$, let $f = T_A u$ and $\lambda = \frac{1}{2|A|} \int_A u d\mu$. Using (4.1), we obtain

$$\|T_A u\|_{p(\cdot)} = \|2\lambda \chi_A\|_{p(\cdot)} \leq 2\|\lambda \chi_{\{f^* > \lambda\}}\|_{p(\cdot)} \leq C \|u\|_{p(\cdot)}.$$

□

Theorem 4.5. *Let $p \in L_+^0(\Omega)$ with $p^- > 0$ and $A \in \Sigma$ with $|A| > 0$, then*

$$(4.2) \quad \|\chi_A\|_{p(\cdot)} \sim |A|^{\frac{1}{p_A}}.$$

Proof of Theorem 4.5. We split the proof into two steps.

Step 1. Let $p \geq 1$. Setting $\varphi_p(t) = t^p$, we regard $\varphi_p(t)$ as a function of two variables p and t . For a fixed $t \geq 0$, $\varphi_p(t)$ is a convex function of p . Defining

$$\varphi_p^{-1}(t) = \inf\{s \geq 0, \varphi_p(s) \geq t\},$$

we have $\varphi_p^{-1}(t) = t^{\frac{1}{p}}$. Let p' be p 's conjugate exponent. Setting $\varphi_{p'}(t) = t^{p'}$, we have $\varphi_{p'}^{-1}(t) = t^{\frac{1}{p'}}$ with $t^\infty = \infty \chi_{(1, \infty)}(t)$ for given $t \geq 0$. It follows that

$$(4.3) \quad \varphi_p(\varphi_p^{-1}(t)) \leq t, \quad \varphi_p^{-1}(t) \varphi_p^{-1}\left(\frac{1}{t}\right) = 1, \quad \varphi_p^{-1}(t) \varphi_{p'}^{-1}(t) \geq t.$$

For $t > 0$ and $|A| > 0$, we have

$$(4.4) \quad \frac{1}{|A|} \int_A \varphi_p^{-1}(t) d\mu \geq \frac{t}{|A|} \int_A \frac{1}{\varphi_{p'}^{-1}(t)} d\mu \geq \frac{t}{\frac{1}{|A|} \int_A \varphi_{p'}^{-1}(t) d\mu},$$

where we have used Jensen's inequality and the convexity of $z \rightarrow \frac{1}{z}$.

Let $u = \varphi_p^{-1}(\frac{1}{|A|}) \chi_A$, $v = \varphi_{p'}^{-1}(\frac{1}{|A|}) \chi_A$. It is easy to check that $\rho_p(u) \leq 1$, $\rho_{p'}(v) \leq 1$. Then $\|u\|_{p(\cdot)} \leq 1$, $\|v\|_{p'(\cdot)} \leq 1$. Applying (2.5), we have

$$\int_A \varphi_{p'}^{-1}\left(\frac{1}{|A|}\right) d\mu = \int_A v d\mu \leq C \|\chi_A\|_{p(\cdot)} \|v\|_{p'(\cdot)} \leq C \|\chi_A\|_{p(\cdot)}.$$

Let $t = |A|$. Then it follows from (4.3) and the convexity of $\varphi_{p'}^{-1}(\frac{1}{t})$ that

$$(4.5) \quad \begin{aligned} |A|^{\frac{1}{p_A}} &= \varphi_{p_A}^{-1}(|A|) \leq \frac{1}{\varphi_{p_A}^{-1}(\frac{1}{|A|})} \leq |A| \varphi_{p_A}^{-1}\left(\frac{1}{|A|}\right) \\ &\leq \int_A \varphi_{p'}^{-1}\left(\frac{1}{|A|}\right) d\mu \leq C \|\chi_A\|_{p(\cdot)}. \end{aligned}$$

On the other hand, using Lemma 4.4, we have

$$\|\chi_A\|_{p(\cdot)} \frac{1}{|A|} \int_A \varphi_p^{-1}\left(\frac{1}{|A|}\right) d\mu = \left\| \frac{1}{|A|} \int_A u d\mu \chi_A \right\|_{p(\cdot)} = \|T_A u\|_{p(\cdot)} \leq C \|u\|_{p(\cdot)} \leq C.$$

It follows from (4.3) and the convexity of $\varphi_p^{-1}(\frac{1}{t})$ that

$$(4.6) \quad \|\chi_A\|_{p(\cdot)} \leq \frac{C}{\frac{1}{|A|} \int_A \varphi_p^{-1}(\frac{1}{|A|}) d\mu} \leq \frac{C}{\varphi_{p_A}^{-1}(\frac{1}{|A|})} \leq C \varphi_{p_A}^{-1}(|A|) = C|A|^{\frac{1}{p_A}}.$$

Combining this with (4.5), we obtain (4.2) for $p \geq 1$.

Step 2. For $p \in L_+^0(\Omega)$ with $p^- > 0$, let $q = p/p^-$. By Step 1, we have $\|\chi_A\|_{q(\cdot)} \sim |A|^{\frac{1}{q_A}}$. It follows from Lemmas 4.2(1) and 2.3(1) that

$$|A|^{\frac{1}{p_A}} = |A|^{\frac{1}{p^- q_A}} = (|A|^{\frac{1}{q_A}})^{\frac{1}{p^-}} \sim \|\chi_A\|_{q(\cdot)}^{\frac{1}{p^-}} = \|\chi_A\|_{p(\cdot)/p^-}^{1/p^-} = \|\chi_A\|_{p(\cdot)}.$$

□

Theorem 4.6. *Let $p, q, r \in L_+^0(\Omega)$.*

(1) *If $p \geq 1$, then*

$$(4.7) \quad \|\chi_A\|_{p(\cdot)} \|\chi_A\|_{p'(\cdot)} \sim |A|,$$

where p' is p 's conjugate exponent;

(2) *If $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ with $r^- > 0$, then*

$$(4.8) \quad \|\chi_A\|_{r(\cdot)} \sim \|\chi_A\|_{p(\cdot)} \|\chi_A\|_{q(\cdot)}.$$

Proof of Theorem 4.6. Using Lemmas 4.4 and 4.2, we have

$$\|\chi_A\|_{p(\cdot)} \|\chi_A\|_{p'(\cdot)} \sim |A|^{\frac{1}{p_A}} |A|^{\frac{1}{p'_A}} = |A|^{\frac{1}{p_A} + \frac{1}{p'_A}} = |A|$$

and

$$\|\chi_A\|_{r(\cdot)} \sim |A|^{\frac{1}{r_A}} = |A|^{\frac{1}{p_A} + \frac{1}{q_A}} = |A|^{\frac{1}{p_A}} |A|^{\frac{1}{q_A}} \sim \|\chi_A\|_{p(\cdot)} \|\chi_A\|_{q(\cdot)}.$$

□

5. SOME INEQUALITIES FOR SPACES WITH $0 < p^- \leq p^+ \leq \eta$

In this section, using the atomic decompositions and the harmonic mean, we prove several martingale inequalities of the Hardy spaces with small exponents $0 < p^- \leq p^+ \leq \eta$, where η is the constant in Theorem 2.5.

Theorem 5.1. *If $p \in \mathcal{P}$ with $p^+ \leq \eta$, then*

$$(5.1) \quad \|f\|_{H_{p(\cdot)}^*} \lesssim \|f\|_{H_{p(\cdot)}^s}, \|f\|_{H_{p(\cdot)}^S} \lesssim \|f\|_{H_{p(\cdot)}^s}, \forall f = (f_n).$$

In particular,

$$\|S^2(f)\|_{p(\cdot)} \lesssim \|s^2(f)\|_{p(\cdot)}, \forall f = (f_n).$$

Proof of Theorem 5.1. Let $f \in H_{p(\cdot)}^s$. It follows from Theorem 3.1 that f has an atomic decomposition satisfying (3.1) and (3.2).

For every $p - A_1$ atom a^k , we estimate the quasi-norms of a^{k*} and $S(a^k)$ in $L^{p(\cdot)}$. For p with $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$, we have $p \leq q \leq 2$. Using (2.5), (3.6), Theorem 4.6(2) and classical Burkholder-Gundy-Davis inequality, we have

$$\begin{aligned}
 \|a^{k*}\|_{p(\cdot)} &= \|a^{k*}\chi_{A_k}\|_{p(\cdot)} \leq C\|a^{k*}\|_2\|\chi_{A_k}\|_{q(\cdot)} \\
 &\leq C\|S(a^k)\|_2\|\chi_{A_k}\|_{q(\cdot)} = C\|s(a^k)\|_2\|\chi_{A_k}\|_{q(\cdot)} \\
 (5.2) \quad &\leq \frac{C\|\chi_{A_k}\|_2\|\chi_{A_k}\|_{q(\cdot)}}{\|\chi_{A_k}\|_{p(\cdot)}} \leq C.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 \|S(a^k)\|_{p(\cdot)} &= \|S(a^k)\chi_{A_k}\|_{p(\cdot)} \leq C\|S(a^k)\|_2\|\chi_{A_k}\|_{q(\cdot)} \\
 &= C\|s(a^k)\|_2\|\chi_{A_k}\|_{q(\cdot)} \\
 (5.3) \quad &\leq \frac{C\|\chi_{A_k}\|_2\|\chi_{A_k}\|_{q(\cdot)}}{\|\chi_{A_k}\|_{p(\cdot)}} \leq C.
 \end{aligned}$$

It follows from (3.1) and (2.6) that $f^* \leq \sum_{k \in \mathbb{Z}} \theta_k a^{k*}$ and

$$\begin{aligned}
 \|f^*\|_{p(\cdot)}^\eta &\leq \sum_{k \in \mathbb{Z}} \|\theta_k a^{k*}\|_{p(\cdot)}^\eta \leq \sum_{k \in \mathbb{Z}} \theta_k^\eta \|a^{k*}\|_{p(\cdot)}^\eta \\
 &\leq C \sum_{k \in \mathbb{Z}} \theta_k^\eta \leq C \left(\sum_{k \in \mathbb{Z}} \theta_k^{p^+} \right)^{\frac{\eta}{p^+}}.
 \end{aligned}$$

Combining this with (3.2), we get

$$\|f^*\|_{p(\cdot)} \leq C\|f\|_{H_{p(\cdot)}^s}.$$

Similarly, we have

$$\|S(f)\|_{p(\cdot)} \leq C\|f\|_{H_{p(\cdot)}^s}.$$

Taking $p/2$ instead of p in the last inequality and using Lemma 2.3(1), we get

$$\|S^2(f)\|_{p(\cdot)} \leq C\|s^2(f)\|_{p(\cdot)}.$$

□

Theorem 5.2. *If $p \in \mathcal{P}$ with $p^+ \leq \eta$, then*

$$(5.4) \quad \|f\|_{H_{p(\cdot)}^*} \lesssim \|f\|_{\mathcal{Q}_{p(\cdot)}}, \quad \|f\|_{H_{p(\cdot)}^s} \lesssim \|f\|_{\mathcal{D}_{p(\cdot)}}, \quad \forall f = (f_n).$$

Proof of Theorem 5.2. Let $f \in \mathcal{Q}_{p(\cdot)}$. It follows from Theorem 3.2 that f has an atomic decomposition satisfying (3.7) and (3.8). For every $p - A_2$ atom a^k , we estimate the quasi-norm of a^{k*} in $L^{p(\cdot)}$. For p with $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$, we have $p \leq q \leq 2$. Using (2.5), Theorem 4.6(2)

and classical Burkholder-Gundy-Davis inequality, we obtain

$$\begin{aligned}
 \|a^{k*}\|_{p(\cdot)} &= \|a^{k*}\chi_{A_k}\|_{p(\cdot)} \leq C\|a^{k*}\|_2\|\chi_{A_k}\|_{q(\cdot)} \\
 &\leq C\|S(a^k)\chi_{A_k}\|_2\|\chi_{A_k}\|_{q(\cdot)} \\
 &\leq C\|\lambda_\infty\chi_{A_k}\|_2\|\chi_{A_k}\|_{q(\cdot)} \\
 (5.5) \quad &\leq \frac{C\|\chi_{A_k}\|_2\|\chi_{A_k}\|_{q(\cdot)}}{\|\chi_{A_k}\|_{p(\cdot)}} \leq C,
 \end{aligned}$$

where λ is f 's optimal predictable control in $\mathcal{Q}_{p(\cdot)}$.

It follows from (3.7) and (2.6) that $f^* \leq \sum_{k \in \mathbb{Z}} \theta_k a^{k*}$ and

$$\begin{aligned}
 \|f^*\|_{p(\cdot)}^\eta &\leq C \sum_{k \in \mathbb{Z}} \|\theta_k a^{k*}\|_{p(\cdot)}^\eta \leq C \sum_{k \in \mathbb{Z}} \theta_k^\eta \|a^{k*}\|_{p(\cdot)}^\eta \\
 &\leq C \sum_{k \in \mathbb{Z}} \theta_k^\eta \leq C \left(\sum_{k \in \mathbb{Z}} \theta_k^{p^+} \right)^{\frac{\eta}{p^+}}.
 \end{aligned}$$

Combining this with (3.8), we get

$$\|f\|_{H_{p(\cdot)}^*} \leq C\|f\|_{\mathcal{Q}_{p(\cdot)}}.$$

Similarly, we get

$$\|f\|_{H_{p(\cdot)}^S} \leq C\|f\|_{\mathcal{D}_{p(\cdot)}}.$$

□

6. SOME INEQUALITIES IN SPACES WITH $0 < p^- \leq p^+ < \infty$

In the last section we extend some martingale inequalities in the spaces with $0 < p^- \leq p^+ \leq \eta$ and the spaces with $1 \leq p^- \leq p^+ < \infty$ to the spaces with $0 < p^- \leq p^+ < \infty$.

Theorem 6.1. *Let $p \in \mathcal{P}$. Then*

$$(6.1) \quad \|f\|_{H_{p(\cdot)}^*} \lesssim \|f\|_{\mathcal{Q}_{p(\cdot)}}, \forall f = (f_n).$$

Proof of Theorem 6.1. We apply an idea due to Chevalier [2] (see also Weisz [24]) and split the proof into two steps.

Step 1. To prove that if (6.1) holds for $p/2$, then it also holds for p . For this purpose, we suppose that

$$(6.2) \quad \|f\|_{H_{p(\cdot)/2}^*} \leq C\|f\|_{\mathcal{Q}_{p(\cdot)/2}}, \forall f \in \mathcal{Q}_{p(\cdot)/2}.$$

Let $f \in \mathcal{Q}_{p(\cdot)}$. We define $g_n = f_n^2 - S_n^2(f)$. Then $dg_n = 2f_{n-1}df_n$ and $g = (g_n)$ is a martingale with

$$S_n(g)^2 \leq 4f_{n-1}^{*2} S_n(f)^2 \leq 4f_{n-1}^{*2} \lambda_{n-1}^2,$$

where (λ_n) is a predictable control of $(S_n(f))$. Using (2.5), we have

$$\|g\|_{\mathcal{Q}_{p(\cdot)/2}} \leq 2\|f^*\lambda_\infty\|_{p(\cdot)/2} \leq 2C_1\|f^*\|_{p(\cdot)}\|\lambda_\infty\|_{p(\cdot)}.$$

Thus

$$(6.3) \quad \|g\|_{\mathcal{Q}_{p(\cdot)/2}} \leq 2C_1\|f^*\|_{p(\cdot)}\|f\|_{\mathcal{Q}_{p(\cdot)}}.$$

Since $f_n^2 = g_n + S_n^2(f)$, we have $|f_n| \leq |g_n|^{\frac{1}{2}} + S_n(f)$. Moreover,

$$(6.4) \quad \|f\|_{H_{p(\cdot)}^*} \leq K(\|g^{\frac{1}{2}}\|_{p(\cdot)} + \|S(f)\|_{p(\cdot)}) = K(\|g\|_{H_{p(\cdot)/2}^*}^{\frac{1}{2}} + \|f\|_{\mathcal{Q}_{p(\cdot)}}).$$

It is clear that $f \in \mathcal{Q}_{p(\cdot)}$ implies $f \in \mathcal{Q}_{p(\cdot)/2}$. Combining (6.2) and (6.3), we get

$$\|g\|_{H_{p(\cdot)/2}^*} \leq C\|g\|_{\mathcal{Q}_{p(\cdot)/2}} \leq 2CC_1\|f^*\|_{p(\cdot)}\|f\|_{\mathcal{Q}_{p(\cdot)}}.$$

It follows from (6.4) that

$$\|f\|_{H_{p(\cdot)}^*} - K(\|f\|_{\mathcal{Q}_{p(\cdot)}} + \sqrt{2CC_1}\|f\|_{H_{p(\cdot)}^*}^{\frac{1}{2}}\|f\|_{\mathcal{Q}_{p(\cdot)}}^{\frac{1}{2}}) \leq 0,$$

Let $z = \|f\|_{H_{p(\cdot)}^*}^{\frac{1}{2}}$. Solving the quadratic inequality of z , we have

$$\|f\|_{H_{p(\cdot)}^*} \leq K^2(2CC_1 + 1)^2\|f\|_{\mathcal{Q}_{p(\cdot)}},$$

where $K^2(2CC_1 + 1)^2$ is a constant depending only on p .

Step 2. Let $0 < p^- \leq p^+ < \infty$. We take positive integer m such that $2^{-m}p^+ \leq \eta$. It follows from Theorem 5.2 that

$$\|f\|_{H_{2^{-m}p(\cdot)}^*} \leq C\|f\|_{\mathcal{Q}_{2^{-m}p(\cdot)}}, \quad \forall f = (f_n).$$

Using Step 1, we have

$$\|f\|_{H_{2^{-m+1}p(\cdot)}^*} \leq C\|f\|_{\mathcal{Q}_{2^{-m+1}p(\cdot)}}, \quad \forall f = (f_n).$$

Then by induction, we have

$$\|f\|_{H_{p(\cdot)}^*} \leq C\|f\|_{\mathcal{Q}_{p(\cdot)}}, \quad \forall f = (f_n).$$

□

Theorem 6.2. *Let $p \in \mathcal{P}$. Then*

$$(6.5) \quad \|f\|_{H_{p(\cdot)}^s} \lesssim \|f\|_{\mathcal{D}_{p(\cdot)}}, \quad \|f\|_{H_{p(\cdot)}^S} \lesssim \|f\|_{\mathcal{D}_{p(\cdot)}}, \quad \forall f = (f_n).$$

Proof of Theorem 6.2. *Step 1.* To prove that if (6.5) holds for $2p$, then it also holds for p . Let $f \in \mathcal{D}_{p(\cdot)}$ and $\lambda = (\lambda_n)$ be f 's optimal predictable control. we define $dg_n = \frac{df_n}{\sqrt{\lambda_{n-1}}}$, $\forall n \geq 1$.

Then

$$g_n = \sum_{i=1}^n \frac{f_i - f_{i-1}}{\sqrt{\lambda_{i-1}}} = \frac{f_n}{\sqrt{\lambda_{n-1}}} + \sum_{i=1}^{n-1} \frac{f_i}{\sqrt{\lambda_{i-1}\lambda_i}}(\sqrt{\lambda_i} - \sqrt{\lambda_{i-1}}),$$

and

$$(6.6) \quad |g_n| \leq 2\sqrt{\lambda_{n-1}}, \quad g^* \leq 2\sqrt{\lambda_\infty}, \quad Eg^{*2} \leq 4E\lambda_\infty < \infty.$$

Thus $g = (g_n)$ is an L^2 -bounded martingale, which converges to $g_\infty = \sum_{i=1}^\infty \frac{df_i}{\sqrt{\lambda_{i-1}}}$ a.e. and

in L^2 . Notice that $df_n = \sqrt{\lambda_{n-1}}dg_n$, $\forall n \geq 1$, then

$$(6.7) \quad s_n^2(f) \leq \lambda_{n-1}s_n^2(g), \quad S_n^2(f) \leq \lambda_{n-1}S_n^2(g).$$

For $s(f)$, by Lemma 2.4 and (2.5), we get

$$\|s(f)\|_{p(\cdot)} \leq \|\lambda_\infty^{\frac{1}{2}} s(g)\|_{p(\cdot)} \leq C \|\lambda_\infty^{\frac{1}{2}}\|_{2p(\cdot)} \|g\|_{H_{2p(\cdot)}^s}.$$

It follows from (6.6) that

$$(6.8) \quad \|s(f)\|_{p(\cdot)} \leq C \|\lambda_\infty^{\frac{1}{2}}\|_{2p(\cdot)} \|g\|_{\mathcal{D}_{2p(\cdot)}} \leq C \|\lambda_\infty^{\frac{1}{2}}\|_{2p(\cdot)}^2 = C \|\lambda_\infty\|_{p(\cdot)}.$$

Thus $\|s(f)\|_{H_{p(\cdot)}^s} \leq C \|f\|_{\mathcal{D}_{p(\cdot)}}$.

Similarly, we have $\|f\|_{H_{p(\cdot)}^S} \leq C \|f\|_{\mathcal{D}_{p(\cdot)}}$.

Step 2. Let $0 < p^- \leq p^+ < \infty$. We take positive integer m such that $2^m p^- \geq 1$. It follows from Theorem [15, Theorem 4.4] that

$$\|f\|_{H_{2^m p(\cdot)}^s} \leq C \|f\|_{\mathcal{D}_{2^m p(\cdot)}}, \|f\|_{H_{2^m p(\cdot)}^S} \leq C \|f\|_{\mathcal{D}_{2^m p(\cdot)}}, \forall f = (f_n).$$

Using Step 1, we have

$$\|f\|_{H_{2^{m-1} p(\cdot)}^s} \leq C \|f\|_{\mathcal{D}_{2^{m-1} p(\cdot)}}, \|f\|_{H_{2^{m-1} p(\cdot)}^S} \leq C \|f\|_{\mathcal{D}_{2^{m-1} p(\cdot)}}, \forall f = (f_n).$$

Then by induction, we have

$$\|f\|_{H_{p(\cdot)}^s} \leq C \|f\|_{\mathcal{D}_{p(\cdot)}}, \|f\|_{H_{p(\cdot)}^S} \leq C \|f\|_{\mathcal{D}_{p(\cdot)}}, \forall f = (f_n).$$

□

Theorem 6.3. *Let $p \in \mathcal{P}$. Then $\mathcal{Q}_{p(\cdot)} \sim \mathcal{D}_{p(\cdot)}$.*

Proof of Theorem 6.3. Let $f = (f_n) \in \mathcal{D}_{p(\cdot)}$ and $\lambda = (\lambda_n)$ be f 's optimal predictable control. Then

$$S_n(f) \leq S_{n-1}(f) + |df_n| \leq S_{n-1}(f) + 2\lambda_{n-1}.$$

It is clear that $(S_{n-1}(f) + 2\lambda_{n-1})_{n \geq 0}$ is a predictable control of $(S_n(f))_{n \geq 0}$ and $f \in \mathcal{Q}_{p(\cdot)}$. It follows from (6.5) that

$$\|f\|_{\mathcal{Q}_{p(\cdot)}} \leq \|S(f) + 2\lambda_\infty\|_{p(\cdot)} \leq K(\|f\|_{H_{p(\cdot)}^S} + 2\|\lambda_\infty\|_{p(\cdot)}) \leq C \|f\|_{\mathcal{D}_{p(\cdot)}}.$$

On the other hand, let $f = (f_n) \in \mathcal{Q}_{p(\cdot)}$ and $\lambda = (\lambda_n)$ be f 's optimal predictable control. Then

$$|f_n| \leq |f_{n-1}| + |df_n| \leq f_{n-1}^* + 2\lambda_{n-1}$$

and $f \in \mathcal{D}_{p(\cdot)}$. Using (6.1), we obtain

$$\|f\|_{\mathcal{D}_{p(\cdot)}} \leq \|f^* + 2\lambda_\infty\|_{p(\cdot)} \leq K(\|f\|_{H_{p(\cdot)}^*} + 2\|\lambda_\infty\|_{p(\cdot)}) \leq C \|f\|_{\mathcal{Q}_{p(\cdot)}}.$$

□

In [24], Weisz proved that if (Σ_n) is regular, then all the spaces H_p^* , H_p^S , H_p^s , \mathcal{D}_p and \mathcal{Q}_p are equivalent when $0 < p < \infty$. Recently, Liu and Wang [15] proved its variable exponent analogue for the case $1 \leq p^- \leq p^+ < \infty$. In the rest of this chapter, we extend the result to the case $0 < p^- \leq p^+ < \infty$. Recall that a martingale $f = (f_n)$ is previsible, if there is a real number $R > 0$ such that

$$(6.9) \quad |df_n|^2 \leq R E_{n-1} |df_n|^2, \quad \forall n \geq 0.$$

Weisz [24, Lemma 2.18] showed that the assumption (6.9) can be defined with the exponent p instead of 2. In addition, Weisz [24, Proposition 2.19] also showed that (6.9) holds for all martingale with the same constant R if and only if (Σ_n) is regular.

Theorem 6.4. *If $p \in \mathcal{P}$ and (Σ_n) is regular, then*

$$H_{p(\cdot)}^* \sim H_{p(\cdot)}^S \sim H_{p(\cdot)}^s \sim \mathcal{D}_{p(\cdot)} \sim \mathcal{Q}_{p(\cdot)}.$$

Proof of Theorem 6.4. In view of Theorem 6.3, we have $\mathcal{D}_{p(\cdot)} \sim \mathcal{Q}_{p(\cdot)}$. Following from Theorems 6.1 and 6.2, we obtain that $\mathcal{Q}_{p(\cdot)} \hookrightarrow H_{p(\cdot)}^*$, $\mathcal{D}_{p(\cdot)} \hookrightarrow H_{p(\cdot)}^s$. By regularity, it is easy to see that $S_n(f) \leq R s_n(f)$ and $S(f) \leq R s(f)$, then $H_{p(\cdot)}^s \hookrightarrow H_{p(\cdot)}^S$. We still need to prove that $H_{p(\cdot)}^S \hookrightarrow \mathcal{Q}_{p(\cdot)}$ and $H_{p(\cdot)}^* \hookrightarrow \mathcal{D}_{p(\cdot)}$. The proofs of $H_{p(\cdot)}^S \hookrightarrow \mathcal{Q}_{p(\cdot)}$ and $H_{p(\cdot)}^* \hookrightarrow \mathcal{D}_{p(\cdot)}$ are similar and we only prove $H_{p(\cdot)}^S \hookrightarrow \mathcal{Q}_{p(\cdot)}$.

To prove $H_{p(\cdot)}^S \hookrightarrow \mathcal{Q}_{p(\cdot)}$. Let $\|S(f)\|_{p(\cdot)} = 1$. By regularity, we have

$$S_n(f) \leq S_{n-1}(f) + |df_n| \leq S_{n-1}(f) + R E_{n-1}|df_n| \leq S_{n-1}(f) + R E_{n-1} S_n(f).$$

Then $(S_n(f))_{n \geq 0}$ has a predictable control and

$$(6.10) \quad \|f\|_{\mathcal{Q}_{p(\cdot)}} \lesssim \|S(f)\|_{p(\cdot)} + R \left\| \sup_{n \geq 0} E_{n-1} S_n(f) \right\|_{p(\cdot)}.$$

Let $1 \leq p^- \leq p^+ < \infty$. Following from the proof of [15, Theorem 4.5], we have

$$(6.11) \quad \left\| \left(\sup_{n \geq 0} E_{n-1} S_n(f) \right) \right\|_{p(\cdot)} \lesssim 1.$$

To finish the proof of the theorem, we prove the following lemma which might be useful in some other circumstances.

Lemma 6.5. *Let $1 \leq p^- \leq p^+ < \infty$ and q be a real number with $0 < q < 1$. Then*

$$\left\| \left(\sup_{n \geq 0} E_{n-1} S_n(f) \right)^q \right\|_{p(\cdot)} \lesssim 1.$$

Proof of Lemma 6.5. Since

$$E_{n-1} S_n(f) = S_{n-1}(f) + E_{n-1}(S_n(f) - S_{n-1}(f)),$$

we have

$$\left(\sup_{n \geq 0} E_{n-1} S_n(f) \right)^q \leq S(f)^q + \left(\sum_{n \geq 0} E_{n-1}(S_n(f) - S_{n-1}(f)) \right)^q.$$

In view of convexity lemma for variable exponent martingales (see [15, Lemma 2.3]), we have

$$\left\| \sum_{n \geq 0} E_{n-1}(S_n(f) - S_{n-1}(f)) \right\|_{p(\cdot)} \lesssim \|S(f)\|_{p(\cdot)} = 1.$$

It follows from Lemma 2.1 that

$$\begin{aligned}
 \rho_{p(\cdot)}\left(\left(\sum_{n \geq 0} E_{n-1}(S_n(f) - S_{n-1}(f))\right)^q\right) &= E\left(\sum_{n \geq 0} E_{n-1}(S_n(f) - S_{n-1}(f))\right)^{qp} \\
 &\leq \left(E\left(\sum_{n \geq 0} E_{n-1}(S_n(f) - S_{n-1}(f))\right)^p\right)^q \\
 &= \rho_{p(\cdot)}^q\left(\sum_{n \geq 0} E_{n-1}(S_n(f) - S_{n-1}(f))\right) \lesssim 1.
 \end{aligned}$$

Using again Lemma 2.1, we have

$$\left\| \sum_{n \geq 0} E_{n-1}(S_n(f) - S_{n-1}(f)) \right\|_{p(\cdot)}^q \lesssim 1.$$

Thus

$$\begin{aligned}
 \left\| \left(\sup_{n \geq 0} E_{n-1} S_n(f) \right)^q \right\|_{p(\cdot)} &\lesssim (\|S(f)^q\|_{p(\cdot)} + 1) \\
 &= (\|S(f)\|_{qp(\cdot)}^q + 1) \lesssim 1.
 \end{aligned}$$

□

Let $0 < p^- < 1$. We replace q and p by p^- and p/p^- in Lemma 6.5 respectively. It follows that

$$\left\| \sup_{n \geq 0} E_{n-1} S_n(f) \right\|_{p(\cdot)}^{p^-} = \left\| \left(\sup_{n \geq 0} E_{n-1} S_n(f) \right)^{p^-} \right\|_{p(\cdot)/p^-} \lesssim 1.$$

Combining this with (6.11) and using (6.10), we have

$$\|f\|_{\mathcal{Q}_{p(\cdot)}} \lesssim 1,$$

where $0 < p^- \leq p^+ < \infty$. Thus $H_{p(\cdot)}^S \hookrightarrow \mathcal{Q}_{p(\cdot)}$. This finishes the proof of the theorem. □

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